Perturbative renormalizability of chiral two-pion exchange in nucleon-nucleon scattering

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We study the perturbative renormalizability of chiral two-pion exchange for singlet and triplet channels within effective field theory, provided that the one-pion exchange piece of the interaction has been fully iterated. We determine the number of counterterms/subtractions needed to obtain finite results when the cutoff is removed, resulting in three counterterms for the singlet channel and six for the triplet. The results show that perturbative chiral two-pion exchange reproduce the data up to a center-of-mass momentum of $k \sim 200$–300 MeV in the singlet channel and $k \sim 300$–400 MeV in the triplet.

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I. INTRODUCTION

The effective field theory (EFT) formulation of nuclear forces [1–4] tries to exploit in a systematic manner the separation of scales between pion physics, which is known to dominate at large distances, and short-range physics in the two-nucleon system. In Weinberg's original proposal [5,6] the chiral nucleon-nucleon potential is organized as a power expansion (or counting) in terms of Q,

$$V_{\rm NN}(r) = V^{(0)}(r) + V^{(2)}(r) + V^{(3)}(r) + \mathcal{O}(Q^4), \qquad (1)$$

where Q represents the low-energy scales of the system, usually the momentum p of the nucleons and the pion mass m_π. The potential is then inserted into the Schrödinger or Lippmann-Schwinger equation to obtain theoretical predictions [7–32]. This prescription is usually referred to as Weinberg's counting.

The resulting chiral potentials turn out to be singular, behaving at order Q^ν as $1/r^{3+\nu}$ in coordinate space for short enough distances ($m_\pi r \ll 1$). Therefore they need to be regularized to obtain well-defined results, usually by introducing a cutoff in the computations plus the necessary number of counterterms that ensure the renormalizability of the scattering amplitude.[1] This has been found to be in contradiction to Weinberg's power counting, where the corresponding counterterms, determined by naive dimensional analysis, are not able to render the theory renormalizable [19–21,26] (or generate chiral inconsistencies [9], prompting the KSW counting [33,34]). Consequently one is forced to make a decision: either to follow an *a priori* power counting or to require renormalizability.

The direct and practical choice is to follow Weinberg's original counting unaltered, leading to a framework amicable with large numerical computations, which demystifies nuclear forces and enjoys an undisputed phenomenological success [15,18]. The price to pay is that the cutoff must be fine-tuned, lying inside a narrow window, a situation that we regard as unsatisfactory from a theoretical viewpoint. Recently, based on the renormalization philosophy of Lepage [35,36], there have been interesting attempts to justify this particular approach [37,38].

In contrast, if one strives for a more robust theoretical foundation, one should be able to achieve cutoff independence. The results from nonperturbative renormalization in the case of singular interactions [19–21,26,39–41] can be summarized as follows: one counterterm is needed to renormalize a channel where the potential is attractive and singular, while channels where the potential is singular and repulsive become insensitive to counterterms. The first condition can lead to an alarming loss of predictive power, as already at leading order (LO) there is an infinite number of attractive singular channels. The solution proposed in Ref. [19] is to treat all partial waves with sufficiently high angular momentum perturbatively at LO, a procedure that is supported by the analysis of Ref. [42].[2] The second condition is particularly problematic: in the triplet channel the potential is attractive at LO but becomes repulsive at next-to-leading order (NLO), resulting in an unbound deuteron at this order when the cutoff is removed [20]. As there is no way to predict what the sign of the interaction will be at higher orders, this represents a continuous threat to the nonperturbative renormalizability of the chiral potentials. In addition, there exists the risk that nonperturbative renormalization of the subleading pieces of the potential may lead to incompatibilities with the chiral expansion [38]. The previous issues can be avoided with the perturbative treatment of the higher order pieces of the potential, which respects power counting and renormalizability independently of whether the subleading contributions are repulsive or attractive. The problem is how to construct such a perturbation theory.

The purpose of this paper is to investigate the conditions under which perturbative chiral two-pion exchange (TPE) can be renormalized to extend the power counting proposal of Nogga *et al.* [19] to subleading orders. In the spirit of Refs. [19] and [43], we use renormalizability as a guide to identify the required short-distance operators. The technical meaning of

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[1] It should be noted, though, that renormalization can be understood in other ways; see the following paragraph.

[2] An alternative solution was recently proposed in Ref. [25].

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renormalizability depends on whether we are in a perturbative or nonperturbative context. By perturbative renormalizability we refer to the elimination of all negative (positive) powers of the coordinate (momentum) space cutoff in the observables. In contrast, nonperturbative renormalizability deals with ambiguities instead of divergences: the scattering amplitude of an attractive singular interaction is finite but nonunique and requires the inclusion of a counterterm for fixing the solution [20,21,39]. A particularly straightforward manner in which to fulfill the renormalization program is to study the cutoff dependence of the amplitudes when the cutoff is removed, as exemplified in Ref. [19]. This should not be interpreted, however, as the necessity to eliminate the cutoff in the computations: after the renormalization process, the residual cutoff dependence of the amplitudes is in principle a higher order effect, provided the cutoff lies within a sensible range.

The perturbative techniques in this paper are based directly on those sketched in Ref. [20] and are equivalent to the momentum space perturbative methods developed in Ref. [43]. Here we use the renormalized distorted-wave Born approximation (DWBA) with the aim of constructing phase shifts. Complementarily, the approach in Refs. [44–47] employs DWBA techniques for "deconstructing" phenomenological phase shifts, that is, for extracting the corresponding short-range physics once the long-range pion effects have been removed and checking whether this short-range interaction is consistent with the specific power counting under consideration, be it Weinberg [44] or Nogga *et al.* [45–47]. Of particular interest is the recent deconstruction of the $^1S_0$ singlet channel [46], which advances some of the results and conclusions of the present work. The present approach differs, however, from the finite cutoff perturbative setup in Ref. [48], in which not all the operators needed to obtain renormalized results are included, as the previous work concentrates on analyzing the Weinberg counting.

The paper is organized as follows: in Sec. II we study the perturbative renormalizability of the $^1S_0$ singlet channel and determine the cutoff and momentum range for which an acceptable description of the data is obtained. We extend the previous results to the the $^3S_1$-$^3D_1$ triplet channel in Sec. III. The role of the cutoff within the present approach is analyzed in Sec. IV, and the relation to other approaches, particularly the renormalization group analysis in Ref. [42], is considered in detail in Sec. V. Finally, we briefly summarize our results in Sec. VI. The technical details of the perturbative treatment of chiral TPE are explained in Appendixes A and B. Some of the $^1S_0$ singlet results in this paper were advanced in Ref. [49].

II. SINGLET CHANNEL

The present perturbative treatment of chiral TPE is based on distorted-wave Born approximation. For simplicity, we only consider the singlet case in detail. We can express the phase shifts as the series

$$\delta(k;r_c) = \delta^{(0)}(k;r_c) + \delta^{(2)}(k;r_c) + \delta^{(3)}(k;r_c) + \mathcal{O}(Q^4), \tag{2}$$

which is ordered according to the counting of the finite-range piece of the potential.[3] That is, power counting is now manifest in the amplitudes. The LO phase shift $\delta^{(0)}$ is computed nonperturbatively (and includes one counterterm),[4] while $\delta^{(2)}$ and $\delta^{(3)}$ are computed in first-order perturbation theory.[5] The corresponding expression for the perturbative phase shifts is (see Appendix A)

$$\frac{\delta^{(\nu)}(k;r_c)}{\sin^2\delta^{(0)}} = -\frac{2\mu}{k}\,\mathcal{A}^{(0)}(k;r_c)^2\,I^{(\nu)}_{^1S_0}(k;r_c), \tag{3}$$

where $\nu = 2, 3$ and the perturbative integral $I^{(\nu)}_{^1S_0}$ is defined as

$$I^{(\nu)}_{^1S_0}(k;r_c) = \int_{r_c}^{\infty} dr\; V^{(\nu)}(r)\,u_k^{(0)^2}(r). \tag{4}$$

In the previous formulas μ is the reduced mass, k is the center-of-mass momentum, $\mathcal{A}^{(0)}$ is a normalization factor, which is taken to be unity at $k = 0$, and $u_k^{(0)}$ is the LO reduced wave function in an energy-independent normalization at the origin (or at the cutoff radius r_c if we are using a finite cutoff). The asymptotic normalization of $u_k^{(0)}$ is determined by $\mathcal{A}^{(0)}(k)\,u^{(0)}(k) \to \sin(kr + \delta^{(0)})/\sin\delta^{(0)}$ for $r \to \infty$.

As can be easily checked, the perturbative integral diverges as $1/r_c^{\nu+2}$ as a consequence of the short-distance behavior of the reduced wave function $u_k^{(0)}(r) \sim 1$ and the potential $V^{(\nu)}(r) \sim 1/r^{\nu+3}$. The divergences can be cured by making the adequate subtractions. Owing to the energy-independent normalization of $u_k^{(0)}$ at the origin, the terms in the k^2 expansion of $u_k^{(0)} = \sum_n u_{2n}^{(0)} k^{2n}$ are progressively less singular, with $u_{2n}^{(0)} \sim r^{2n}$ for $r \to 0$. Expanding the previous integrals in terms of k^2 for $\nu = 2, 3$, we have

$$I^{(\nu)}_{^1S_0}(k;r_c) = I^{(\nu)}_0(r_c) + k^2\,I^{(\nu)}_2(r_c) + k^4\,I^{(\nu)}_4(r_c) + I^{(\nu)}_{^1S_0,R}(k;r_c), \tag{5}$$

where $I^{(\nu)}_{0,2,4}$ are the divergent pieces of the integral and $I^{(\nu)}_{^1S_0,R}$ is the regular piece, as can be trivially checked. Therefore three subtractions or counterterms are needed to renormalize the perturbative results in the singlet. The specific method employed is not important. Here we modify the perturbative integral by adding three free parameters that are to be fitted to the data:

$$\hat{I}^{(\nu)}_{^1S_0}(k;r_c) = \lambda_0^{(\nu)} + \lambda_2^{(\nu)}k^2 + \lambda_4^{(\nu)}k^4 + I^{(\nu)}_{^1S_0}(k;r_c). \tag{6}$$

By assuming the short-range physics to be parametrized by an energy-dependent δ-shell potential of the type

$$V_C^{(\nu)}(r;r_c) = \frac{\mu}{2\pi\,r_c^2}\sum_n C_{2n}^{(\nu)}(r_c)k^{2n}\,\delta(r - r_c), \tag{7}$$

[3] Strictly speaking, the leading-order piece is of order Q^{-1}, not Q^0. However, to keep the notation simpler, we have just followed Eq. (1).

[4] Note that we do not consider chiral symmetry breaking terms separately here.

[5] Second-order perturbation theory is not needed, as the iteration of the NLO potential is of order Q^5.

we can easily relate the $\lambda_{2n}^{(\nu)}$ parameters to the $C_{2n}^{(\nu)}$ counterterms by

$$\lambda_{2n}^{(\nu)} = \frac{\mu}{2\pi\, r_c^2}\, C_{2n}^{(\nu)}(r_c) u_0^{(0)2}(r_c). \tag{8}$$

Equivalently, if one chooses to work in the momentum space formulation of Ref. [43], one could include the contact potential $\langle p|V_C^{(\nu)}|p'\rangle = C_0^{(\nu)} + C_2^{(\nu)}(p^2+p'^2) + C_4^{(\nu)}(p^4+p'^4)$. In either case, the first free parameter, $\lambda_0^{(\nu)}$ ($C_0^{(\nu)}$), is only used to absorb the $k=0$ divergence of the perturbative integral, while its finite piece is redundant, as it only affects the zero-energy behavior of the phase shifts, which has already been fixed at LO, meaning that we need to fix two additional observables, for example, the effective range r_0 and the shape parameter v_2, to determine the NLO/N^{2}LO results. The number of counterterms agrees with the corresponding one predicted in the renormalization-group analysis (RGA) in Ref. [42], where the power counting resulting from treating one-pion exchange (OPE) nonperturbatively was analyzed in detail, and with the related deconstruction in Ref. [46], in which the short-range physics for the singlet channel is determined by removing the nonperturbative OPE and perturbative TPE effects from the phenomenological phase shifts. Note that in Ref. [20] an incorrect number of counterterms was determined, owing to an improper normalization.

The results for the singlet $^1S_0$ channel are shown in Fig. 1. Following Refs. [16] and [17], we take $f_\pi = 92.4$ MeV, $m_\pi = 138.03$ MeV, $g_A = 1.26$, and $d_{18} = -0.97$ GeV^2. For the chiral couplings we employ the customary values $c_1 = -0.81$ GeV^{-1}, $c_3 = -3.40$ GeV^{-1}, and $c_4 = 3.40$ GeV^{-1}, which are compatible with the determination in Ref. [52]. The potential is taken from Ref. [53]. As can be seen, the

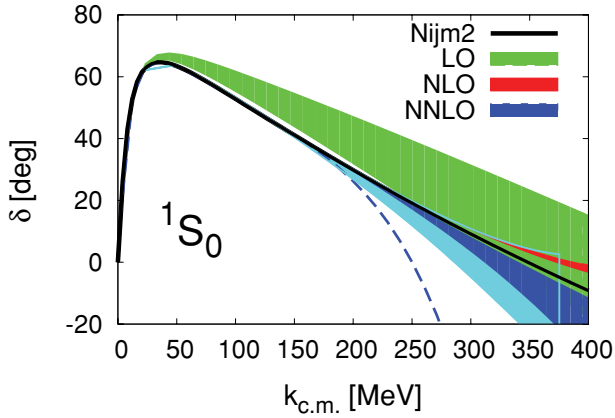

FIG. 1. (Color online) Phase shifts for the $^1S_0$ channel with nonperturbative OPE and perturbative TPE. The nonperturbative OPE computation contains one counterterm, which is determined by fixing the $^1S_0$ scattering length, $a_{0,s} = -23.74$ fm, while the perturbative TPE computation contains a correction to the LO counterterm plus two additional counterterms, which are used to fit the Nijmegen II phase shifts [50] (equivalent to the Nijmegen PWA [51]) in the range $k = 40$–160 MeV. Error bands are generated varying the cutoff within the range 0.6–0.9 fm. The light-blue band represents the N^{2}LO results from the standard Weinberg approach of Ref. [17]. The dashed dark-blue line represents the N^{2}LO results for $r_c = 0.1$ fm.

results reproduce the $^1S_0$ phase shifts up to $k \sim 200$–300 MeV, depending on the value of the cutoff. If the cutoff is small ($r_c = 0.1$ fm), the perturbative treatment of the subleading pieces of the interaction starts to fail already at $k \sim 200$ MeV, as a consequence of the relative weakness of OPE with respect to the enormous strength of TPE at short distances. The previous problems can be circumvented by using cutoffs of the order of $r_c \sim 1/2m_\pi$ (0.7 fm), which are small enough to guarantee the correct inclusion of the TPE tail. In particular, we employ $r_c = 0.6$–0.9 fm,[6] a range for which perturbative TPE calculations compete well with nonperturbative ones in the Weinberg counting at the same order [13,17], though perturbative TPE is slightly less predictive, owing to the additional counterterm. On different grounds, it should be noted that OPE is perturbative in the singlet [33,34,54], even if iterated [55], suggesting that the previous results could be reinterpreted as an N^{3}LO/N^{4}LO computation in the KSW counting [33,34].

The failure of perturbative subleading TPE at $r_c = 0.1$ fm raises interesting questions regarding the adequacy of the present power counting scheme and the role of chiral TPE. Of course, the technical reasons why perturbation theory fails already at $k \sim 200$ fm for small cutoffs are clear: OPE does not provide enough long-range distortion to avoid higher momentum waves to probe the van der Waals component of TPE, as discussed, for example, in Ref. [46]. This component originates from the behavior of subleading TPE, which, in the singlet channel, can be schematically written as [53]

$$2\mu\, V_{\rm TPE}^{(\nu=3)}(r) = -\frac{R_6^4}{r^6}\, e^{-2m_\pi r} \sum_{n=0}^{5} a_n (2m_\pi r)^n, \tag{9}$$

where the a_n's are dimensionless parameters with $a_0 = 1$, and R_6 is a length scale related to the strength of TPE at short distances, which varies between $R_6 = 1.6$ fm and $R_6 = 1.8$ fm for typical values of the chiral couplings. The previous form implies that the chiral van der Waals component of subleading TPE should start to become apparent at distances below $r \leqslant 1/2m_\pi \simeq 0.7$ fm. This figure is supported by several renormalized nonperturbative TPE computations in the singlet [20,21,24], which usually reach cutoff independence at distances around or below 0.5 fm, signaling the onset of chiral van der Waals forces. For such cutoff radii the perturbative treatment of TPE generates terms like kR_6 and $m_\pi R_6$, which, taking into account the size of R_6, might cause the perturbative series to eventually diverge. The most consistent and straightforward solution to this problem is the use of large enough cutoffs ($r_c > 0.5$ fm) to avoid the conjectured breakdown of the perturbative series. The alternative solution, which is not considered in the present work, is the iteration of chiral TPE, or at least some parts of it [46]. Although interesting, this proposal seems difficult to harmonize within the EFT framework, as it requires (i) justification of the promotion of an order Q^3 interaction to order Q^{-1} and

[6]Taking into account the relationship $\Lambda = \pi/2r_c$ [26], the previous configuration space cutoff range is approximately equivalent to a momentum space (sharp) cutoff of $\Lambda \simeq 350$–500 MeV.

(ii) the existence of a cutoff window for which subleading TPE dominates but the higher order corrections are still small compared to this contribution.

The employed cutoff window, $r_c = 0.6$–0.9 fm, represents a compromise between the requirements of singlet and those of triplet channels. The optimum value of the cutoff in the singlet lies in the vicinity of $r_c = 0.9$–1.0 fm, a range for which the description of the triplet phases starts to worsen. This cutoff window may look soft, but it is not: the first deeply bound state (i.e., the first 0 of the $k = 0$ wave function) for the N^{2}LO potential happens at $r_c = 0.70$ fm, meaning that the lower range of the present cutoff window is already beyond what can be reached in the Weinberg scheme. It is interesting to note that the previous cutoff range is similar to the radii at which most potential models of the NN interaction [50,56,57] have their minima, usually at $r \sim 0.8$–0.9 fm. The mimima mark the distance at which the short-range repulsion starts to overcome the long-range attraction and, consequently, can be understood as the separation point between short-range ($r \lesssim 0.5$ fm) and long-range ($r \gtrsim 1.0$ fm) physics. In this sense, the cutoff is to be interpreted as a separation scale, as has been proposed within the context of the RGA [42,58,59], rather than as a hard scale [37,38].

In the calculations in Fig. 1 we also interpret the cutoff variation of the results as the error band of the theory. The previous is a sensible prospect in the sense that we expect the cutoff dependence of the scattering amplitudes to be a higher order effect. However, if the cutoff variation is to be understood as an error band, the size of the band should decrease at each new order to reflect the convergence properties of the theory. Paradoxically the N^{2}LO band is bigger than the NLO one, a worrisome situation that does not necessarily mean that we should abandon the previous interpretation. In fact the same happens in the Weinberg counting, as illustrated by the singlet-channel results in Ref. [17]. The explanation is to be found in the surprisingly large size of the c_3 and c_4 chiral couplings, which causes the subleading TPE contribution to the chiral potential to be substantially bigger than the corresponding one from the leading TPE. This is caused by the large contributions from the Δ resonance to the chiral couplings [60], $c_{3,\Delta} = -2c_{4,\Delta} = -4h_A^2/9\Delta$, with Δ the nucleon-Δ mass splitting and h_A the $\pi N \Delta$ axial coupling, ranging from -1.7 to -2.7 GeV^{-1}, depending on the value of h_A.[7] In this sense, the increased size of the N^{2}LO error bands is just a reflection of the unexpected contribution from this low-energy scale. The explicit inclusion of the Δ resonance in the NN chiral potential, a theme that has been recurrently considered in the literature [8,11,22,24–26], is presumed to solve the current issue with the error bands (see also the related discussion in Ref. [46]). This prospect does not appear to be unreasonable in view of the perturbative peripheral wave N^{2}LO-Δ results in Ref. [22] and the related nonperturbative central and peripheral wave calculations in Refs. [24] and [25], all of which indicate an enhancement in the convergence rate of the phase shifts compared to the Δ-less theory.

III. TRIPLET CHANNEL

In the case of the $^3S_1$-$^3D_1$ channel the perturbative analysis is analogous to the previous one for the $^1S_0$ channel, but more cumbersome, owing to the presence of coupled channels and the singular behavior of the tensor piece of the LO potential in the triplet channels. The details of this analysis are reported in Appendix B, but the essential point is that the inverse power law behavior of the OPE tensor force ($\sim 1/r^3$) softens the perturbative integrals and reduces the necessary number of counterterms per phase. In fact we have that the s- and d-wave functions behave as $u_k^{(0)}$, $w_k^{(0)} \sim r^{3/4}$ near the origin [61] and that each subtraction adds an $r^{5/2}$ suppression to the short-distance behavior.[8] This translates into two subtractions for each of the three phases in the $^3S_1$-$^3D_1$ channel ($\delta_{^3S_1}, \epsilon_1, \delta_{^3D_1}$), meaning that we end up with six counterterms at NLO/N^{2}LO, in agreement with Ref. [20]. That is, the scattering amplitude can be completely determined using six data, for example, the value of the three phase shifts at two different momenta.

The results are shown in Fig. 2. Perturbative TPE provides a good description of the $^3S_1$ and $^3D_1$ phases and the ϵ_1 mixing angle up to moderately high momenta, around $k \sim$ 300–400 MeV, although it should be noted that the results are quite sensitive to the choice of chiral couplings, owing to the linear dependence generated by treating chiral TPE perturbatively. Contrary to the singlet case, small cutoffs do not affect the momentum range in which first-order perturbation theory works, although, owing to numerical limitations, the cutoff cannot be reliably reduced below $r_c = 0.3$ fm. However, there are reasons for keeping the cutoff in the proposed window, such as avoiding unphysical deeply bound states in the LO amplitudes (the first one appears at $r_c = 0.45$ fm), or an excessive D-state probability in the deuteron, yielding poor convergence in nuclear matter calculations [62]. Larger cutoffs, of the order of 1 fm and above, are also disfavored, as they lead to a worse description of the ϵ_1 mixing angle for momenta above 300 MeV, similar to the one obtained in the N^{2}LO Weinberg calculation in Ref. [17]. The proposed cutoff range avoids the previous problems and, owing to the stronger long-range distortion provided by the tensor component of OPE, generate error bands that decrease in size order by order.

In this regard, it is interesting to note the opposite cutoff preferences of the singlet and triplet channels. The mismatch in the preferred cutoff windows is a reflection of the different physics at play in these waves. In the singlet, all pion exchanges are perturbative and the iteration of OPE is merely a shortcut to avoid the computation of higher order perturbations, while in

[7] We take h_A between 1.08 and the SU(4) value 1.34; see Ref. [22]. Values of the chiral couplings once Δ has been included can also be found in Ref. [22].

[8] This is to be compared with the singlet channel, where $\tilde{u}_k^{(0)\,2} \sim 1$ and each subtraction adds an additional r^2 suppression. If the singlet leading order (LO) potential had behaved as expected by power counting, that is, $1/r^3$, it would have needed only two counterterms at NLO/N^{2}LO, following Weinberg's counting.

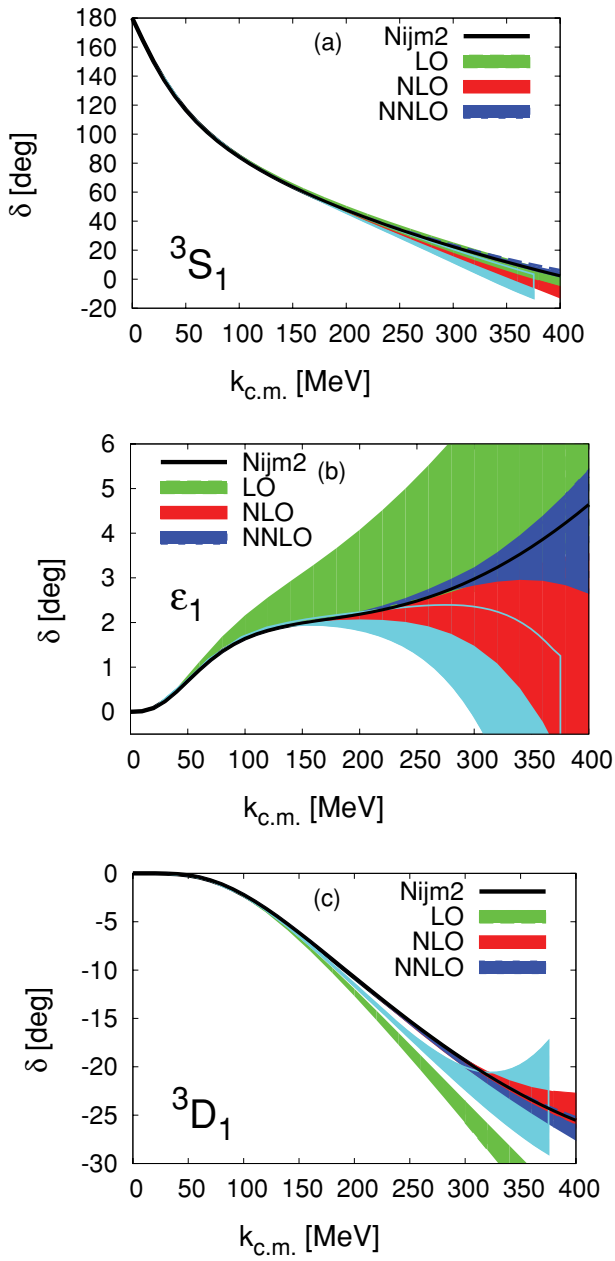

FIG. 2. (Color online) Phase shifts (nuclear bar) for the $^3S_1$-$^3D_1$ coupled channel, with the LO piece (OPE) fully iterated and the NLO and N^{2}LO pieces (chiral TPE) treated perturbatively. The LO counterterm is fixed to reproduce the triplet scattering length $a_{0,t} = 5.419$ fm. Error bands and fitting range are the same as in Fig. 1. The light-blue band corresponds to the N^{2}LO results in Ref. [17] in the standard Weinberg counting. The dashed dark-blue line represents the N^{2}LO results for $r_c = 0.3$ fm.

the triplet tensor OPE really needs to be iterated. Different power countings require different cutoff windows. In this sense, large cutoff values worsen the convergence of the triplet: the OPE tensor force starts to behave perturbatively, even if fully iterated in the Schrödinger equation. This entails a change in the counting of the triplet channel from the modified Weinberg scheme of Nogga *et al.* [19] eventually to KSW [33,34], thus reducing the convergence of the theory as the cutoff is increased. The exact point at which the change takes place is difficult to determine but probably lies above $r_c \gtrsim 1/m_\pi = 1.4$ fm. Of course, the fact that the ϵ_1 mixing angle is the phase that starts to feel the problem earlier is not a surprise, as it depends on delicate cancellations between short- and long-range effects. The singlet channel, on the contrary, does not have any problem with larger values of the cutoff, as the power counting is not changed: the use of larger cutoff values only entails a rearrangement of the short-range physics to account for those parts of the pion tail that have been ignored, but the assumption that all long-range interactions are perturbative remains unchanged.

IV. THE ROLE OF THE CUTOFF

In the previous calculations we have taken a very pragmatic point of view with regard to the cutoff: we have chosen the cutoff range $r_c = 0.6$–0.9 fm to improve the convergence of the theory and the description of the phase shifts up to N^{2}LO. Of course, the choice of this range depends on a compromise between the specific requirements of the singlet and those of the triplet channels, as explained in previous sections. The important point, however, is that the proposed cutoff window generates LO phase shifts that do not differ too much from the Nijmegen ones, an arrangement that minimizes the size of the subleading corrections and, as a consequence, enhances the convergence of the theory.

This criterion basically coincides with the interesting cutoff philosophy of Beane *et al.* [63], in which the cutoff is merely a parameter controlling the convergence rate of the theory. The underlying idea behind this interpretation is an analogy with the role of renormalization scale dependence in QCD (see, e.g., Refs. [64] and [65]). A similar rationale can be provided by the observation that the full scattering amplitude, computed at all orders, is cutoff independent as a consequence of having an infinite number of counterterms. In this regard, cutoff dependence is just an artifact of finite-order approximations, which can be avoided by the careful selection of a cutoff window for which the particular power counting under consideration is realized.

However, for this interpretation to be complete within an EFT context it is necessary first to determine some formal aspects of the theory, such as the expansion parameter and the cutoff and momentum ranges for which the perturbative expansion converges. The knowledge of the expansion parameter is fundamental to enable making rigorous error estimations of the results and checking the suitability of the selected cutoff window. In contrast, the determination of the range of applicability of the EFT is necessary for avoiding power counting abuse, that is, claiming as legitimate the accidental description of data beyond the possibilities of the EFT under consideration.

Owing to the mostly numerical nature of the present investigation, it is not clear how to extract the expansion parameter. However, the deconstruction of the $^1S_0$ singlet channel of Ref. [46] might provide some valuable clues regarding this important aspect of the theory. The energy dependence of the short-range physics in this channel suggests a breakdown

scale of $\Lambda_{0,s} \simeq 270$ MeV. This estimation translates into an expansion parameter of $m_\pi/\Lambda_{0,s} \simeq 0.5$, a value that is compatible with the conjectured equivalence of the present approach with the KSW counting in the singlet channel. For the $^3S_1$-$^3D_1$ triplet channel there is no deconstruction yet that might provide a preliminary estimation of the breakdown scale, but if we assume the deconstruction of the p-wave uncoupled triplets [45] to hold in the $^3S_1$-$^3D_1$ coupled triplet, we obtain $\Lambda_{0,t} \simeq 340$ MeV.[9] The related expansion parameter would be $m_\pi/\Lambda_{0,t} \simeq 0.4$, a value that is compatible with the observation that the convergence is better in the triplet than in the singlet.

V. RELATION TO OTHER APPROACHES

In the present work we determine the power counting of the counterterms by requiring the renormalizability of the perturbative corrections to the scattering amplitude, where renormalizability is understood to be the elimination of all negative powers of the coordinate space cutoff r_c. There is still a residual cutoff dependence that is nominally of higher order, meaning that perturbative renormalizability implies the cutoff independence of the scattering amplitude at the order considered.

This is very similar to the renormalization-group approach of Birse [58], where the relative scaling (i.e., the power counting) of the counterterms is determined by requiring the cutoff independence of the scattering amplitude. Of course, exact cutoff independence is only achieved at infinite order. Finite-order truncations will lead to a residual cutoff dependence involving positive powers of the cutoff r_c, but the renormalizability of the amplitudes is guaranteed. Therefore, the great degree of agreement between these two approaches is not surprising.

This expectation is realized in the singlet channel, where RGA [42] and deconstruction [46] are equivalent to perturbative renormalizability. For the triplet channel the situation is mixed: in the case of the ϵ_1 mixing angle and the $^3D_1$ phase, the observation that two counterterms are needed to renormalize each of these phases is compatible with the deconstruction of the p- and d-wave uncoupled triplets of Ref. [45]. However, the RGA of Ref. [42] predicts one additional counterterm for the $^3S_1$ phase, which should appear at order $Q^{5/2}$. This counterterm is not needed by perturbative renormalizability.

The previous discrepancy is surprising: we are making the same assumptions as in Ref. [42] regarding which pieces of the interaction to iterate, yet the resulting power countings are slightly different. However, this is not new: the nonperturbative renormalizability of the OPE potential dictates that each attractive triplet requires one counterterm, while repulsive triplets do not. On the contrary, the RGA of Ref. [42] makes no distinction for the power counting of attractive and repulsive triplets. The paradigmatic example is given by the $^3P_0$ (attractive) and $^3P_1$ (repulsive) waves. As happens in peripheral waves, the inconsistency can be circumvented in terms of the perturbative analysis of tensor OPE in Ref. [42]: for the $^3P_0$ wave the perturbative treatment of OPE is expected to fail already at $k \sim 200$ MeV, while for the $^3P_1$ wave this limit is extended up to $k \sim 400$ MeV. Therefore, in the range of momenta of interest for nuclear EFT, the $^3P_1$ wave can in principle be described in terms of the original Weinberg counting.[10]

For the $^3S_1$ phase the causes of the disagreement are to be found in the naive extrapolation of the idea of trivial and nontrivial fixed points to a problem where these concepts may not be applicable. The relevant observation in this context is that attractive singular potentials do not have a unique solution [20,21,39]: the value of the scattering length oscillates indefinitely as the cutoff varies, a situation that is solved by the inclusion of a counterterm, stabilizing the solution. In this regard, for an attractive singular interaction all values of the scattering length are equally fine-tuned, implying that the distinction between trivial and nontrivial fixed points is artificial in this case. As analyzed in Ref. [40], the renormalization-group evolution of attractive singular potentials is driven in the infrared limit[11] toward an oscillatory attractor-type solution resembling a limit cycle. However, the attractor-type solution does not have the discrete scaling properties of limit cycles (see Ref. [40] for details).

Alternatively, the previous observations can also be understood in terms of the behavior of the squared reduced wave functions at short distances. For regular potentials there are two possible behaviors, the regular one, $|u(r_c)|^2 \sim r_c^2$, which can be identified with "natural" systems, and the irregular one, $|u(r_c)|^2 \sim 1$, which describes systems with unnaturally large scattering lengths. On the contrary, for an attractive singular potential, the wave function always behaves as $|u(r_c)|^2 \sim r_c^{3/2}$ (times an oscillatory factor), independently of the value of the scattering length. That is, there is no additional short-range enhancement owing to the large scattering lengths. In this regard, we should not expect the existence of two

[9] This corresponds to a laboratory energy of 250 MeV, above which the short-range interaction in the $^3P_0$, $^3P_1$, and $^3D_2$ waves cannot be reliably described by two counterterms [45]. In contrast, the assumption that the p waves yield a good approximation for the breakdown scale of the$s$ waves is not unreasonable if we take into account that the deconstruction of the $^1P_1$ wave [47] basically suggests the same estimation as the $^1S_0$ wave [46].

[10] One could think of extending this argument to the $^3D_1$ phase, which is usually well reproduced in perturbation theory [10]. However, taking into account the coupled-channel nature of the $^3D_1$ phase, it is probably inconsistent to treat tensor OPE perturbatively in the d-wave channel but not in the s-wave channel.

[11] Note, however, that Ref. [40] uses a different language than Birse's RGA [42]: what is called the ultraviolet (long-range) limit in Ref. [40] corresponds to the infrared (short-range) limit in Ref. [42]. If we call the light and heavy scales m_l and m_h, Ref. [40] takes $m_l r_c \to 0$, while Ref. [42] assumes $m_h \gg 1/r_c \gg m_l$ or, equivalently, $m_h r_c \to \infty$, $m_l r_c \to \infty$, and $m_l/m_h \to 0$. Contrary to Ref. [42], Ref. [40] does not analyze the power counting of the short-range operators but, rather, concentrates on issues such as the cutoff dependence of observables and the fixed points, limit cycles, and attractors that result from the renormalization-group flow of regular and singular potentials.

different kinds of fixed points in the RG flow of attractive singular interactions. The previous observations indicate that for attractive singular potentials (i) the C_0 counterterm must be of order Q^{-1}, as required by nonperturbative renormalizability, and (ii) the first perturbation to the C_0 counterterms is of order $Q^{-1/2}$ as expected from the behavior of the squared wave function, meaning that the attractor is a stable solution of the RG flow. Consequently, the correct RG analysis for channels with an attractive tensor force is the one termed "trivial" in Ref. [42], conveniently modified to incorporate the previous observation about the C_0 operator.

A recent work that is also relevant for the present approach is the toy model proposed by Epelbaum and Gegelia to address the role of regularization and renormalization in EFT [38]. In that work, the authors consider a two-body potential problem that shares many of the features of nuclear EFT, like the existence of a separation of scales and the possibility of expanding the long-range interaction in terms of a power counting. The conclusions of the analysis by Epelbaum and Gegelia support most of the assumptions usually invoked in the Weinberg scheme, namely, that naive dimensional analysis provides a good enough power counting and the ideal value of the cutoff should be chosen of the order of the hard scale of the problem. In addition, if the cutoff is taken much beyond the hard scale, the nonperturbative renormalization procedure may break the assumptions made in the first place by the power counting, a phenomenon that Epelbaum and Gegelia call "peratization."

The lessons derived from a specific toy model may be of limited significance, however. In particular, there is an essential feature of the chiral expansion that is not reproduced in the previous model, namely, the appearance of singular interactions at leading and subleading orders. Contrary to the expectations of Epelbaum and Gegelia, the presence of singular potentials implies that (i) nonperturbative power counting will break down at cutoffs much softer than expected and (ii) deviations from naive dimensional analysis may eventually happen. These aspects have probably been overlooked in the previous analysis owing to the very good properties of the toy model: subleading contributions to the toy potential are only mildly divergent, and in addition, they are always suppressed by the expected ratio of low-energy to high-energy scales. On the contrary, the subleading pieces of the chiral NN potential can receive unexpectedly large contributions from light degrees of freedom that have not been explicitly taken into account, like the Δ resonance. It is not surprising therefore that a toy model incorporating many of our naive expectations about EFT turns out to confirm them.

However, as long as we limit ourselves to soft enough cutoffs, the conclusions of Epelbaum and Gegelia regarding naive dimensional analysis (i.e., Weinberg counting) are likely to hold. This observation is realized in the work of Shukla *et al.* [48], which, much in the spirit of deconstruction, analyzes the short-distance physics of the $^1S_0$ singlet channel with the chiral NN potential up to N^{2}LO. The authors observe that in the cutoff region $r_c = 1.0$–1.8 fm, two counterterms are enough to parametrize the short-range physics, a finding consistent with the idea that the Weinberg counting is better realized for soft values of the cutoff. A particularly interesting aspect of the previous work is the reanalysis of the short-range physics for perturbative chiral TPE. For the cutoff range $r_c = 1.4$–1.8 fm the extracted short-range physics can be accurately approximated by first-order perturbative TPE, while for the region $r_c = 1.0$–1.4 fm one needs to go to second and third order in the perturbative series to reproduce the nonperturbative results, although there is still convergence. In the softer cutoff range the Weinberg scheme is perfectly realized as a perturbative power counting. For the harder cutoff range, Weinberg is still a consistent (nonperturbative) power counting scheme, as subleading order corrections are smaller than LO ones. The efforts in Ref. [48] probably represent the best way to analyze the merits of the Weinberg counting in realistic cases. The extension to other partial waves, in particular, the triplet, would be very welcomed.

If the cutoff is decreased below $R_0 = 1.0$ fm, the authors of Ref. [48] observe that the contributions from subleading TPE start to grow uncontrollably, signaling the breakdown of the Weinberg counting. Below this cutoff, power counting is likely to be lost in nonperturbative calculations, as loop contributions from the subleading pieces will eventually dominate the amplitudes. The previous breakdown scale is, however, uncomfortably soft: using the equivalence $\Lambda = \pi/2r_c$ [26], R_0 naively corresponds to a (sharp) momentum cutoff of $\Lambda_0 \simeq 310$ MeV. Most Weinberg calculations use momentum space cutoffs of the order of $\Lambda \sim 0.5$ GeV, which may be hard enough to "peratize" the amplitudes (as meant in Ref. [38]). As suggested in Ref. [49], this may already be happening in the $^1S_0$ singlet channel for $\Lambda = 400$ MeV at N^{2}LO. These observations do not imply, however, that Weinberg counting is not useful: only that it should be employed within its specific range of applicability. In this respect, the most interesting feature of perturbative treatments is that they are guaranteed to respect the power counting independently of the value of the cutoff, precluding from the start the possibility of any power counting inconsistency.

A recent work that is also relevant for the discussion is the new KSW expansion of Beane *et al.* [63], which challenges one of the key premises of the present approach, namely, that OPE should be fully iterated in the triplet, by constructing a viable nuclear EFT in which all pion exchanges are treated as perturbations. In this work the convergence problems of the original KSW counting [42,55] are alleviated by the exchange of a fictitious meson of mass λ that regulates the $1/r^3$ singularity of the tensor force at short distances. For the optimum value of the regulator ($\lambda = 750$ MeV), the expansion apparently converges up to order Q, albeit slowly. At this order, the results of Ref. [63] for the $^3S_1$ and $^3D_1$ phases compare well with the LO results of the present approach. However, the order Q results for the ϵ_1 mixing angle is clearly worse than our LO computation and it does not seem to converge for $k > m_\pi$. This may be a good indicator that the tensor force really needs to be iterated, as the ϵ_1 mixing angle is very sensitive to large cancellations between long- and short-range physics. In any case, a serious comparison of the present approach with the proposal of Beane *et al.* [63] requires (i) extension of the previous results beyond order Q and (ii) consideration of the $^3P_0$ phase, which, according to Nogga *et al.*

[19], also demands the nonperturbative inclusion of tensor OPE.

The observation that OPE is perturbative in the singlet and nonperturbative in the triplet is closely related to the proposal of Beane, Bedaque, Savage, and van Kolck (BBSvK) [54], which suggested the iteration of those pieces of the (LO) chiral NN potential that survive in the chiral limit (i.e., tensor OPE). This prescription is theorized to generate a convergent expansion of the scattering amplitudes around the chiral limit, therefore providing a consistent EFT expansion for two-nucleon systems. The existence of a deeper relationship with the present approach remains to be seen. However, the consideration of the subleading orders of the potential can break the correspondence, as there are pieces of these contributions to the potential that survive in the chiral limit and that are strong enough as to be iterated, particularly in the singlet channel. In this regard, the BBSvK scheme might provide a justification for the iteration of chiral van der Waals forces.

VI. CONCLUSIONS

The present approach determines the momentum and cutoff range for which chiral TPE behaves perturbatively when renormalizability is imposed. The use of small cutoffs is straightforward but reduces the range of applicability of the theory in the singlet channel. The calculations turn out to confirm the viability of the counting proposal of Nogga *et al.* [19] and corroborate, to a large extent, the related RGA of Birse [42], which predicted the power counting of short-range operators. There are some minor discrepancies, however, between perturbative renormalizability and RGA in the triplet channel, specifically for the $^3S_1$ phase, which are understood, suggesting minor modifications and possible improvements to the RGA of Ref. [42]. However, there are some formal aspects of the present EFT formulation that need to be elucidated, like the role of the cutoff, the determination of the expansion parameter, and the range of applicability of perturbative TPE. In this regard, the deconstruction approach of Refs. [45–47] is able to provide some interesting clues and preliminary answers. Of course, a complete evaluation of the renormalized perturbative treatment of chiral TPE should also include the calculation of the p- and d-wave phase shifts and the deuteron properties. The present analysis paves the way for such computations, which we leave for future works.

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APPENDIX A: DERIVATION OF THE DWBA FOR THE SINGLET CHANNEL

In this appendix, we derive the DBWA formulas used throughout the present paper. We start by considering a potential that can be decomposed as a zeroth-order approximation and a perturbation,

$$V(r) = V^{(0)}(r) + V^{(1)}(r), \tag{A1}$$

and the related reduced Schrödinger equations for the zeroth-order and full reduced wave functions, $u_k^{(0)}$ and u_k,

$$-u_k^{(0)\prime\prime} + 2\mu\, V^{(0)}\, u_k^{(0)} = k^2\, u_k^{(0)}, \tag{A2}$$

$$-u_k{}'' + 2\mu\,[V^{(0)} + V^{(1)}]\, u_k = k^2\, u_k, \tag{A3}$$

where μ is the reduced mass of the system. The full reduced wave function can be perturbatively expanded as

$$u_k^{(0+1)}(r) = u_k^{(0)}(r) + u_k^{(1)}(r) + \mathcal{O}((V^{(1)})^2), \tag{A4}$$

where, for the purposes of this work, it is enough to consider first-order perturbation theory only.

To obtain the DWBA expressions we begin by (i) multiplying the zeroth-order Schrödinger equation by the full solution u_k and (ii) multiplying the full Schrödinger equation by the zeroth-order solution $u_k^{(0)}$. Then we compute the difference between (i) and (ii), yielding

$$\left(u_k^{(0)}\, u_k' - u_k^{(0)\prime}\, u_k\right)' = 2\mu\, V^{(1)}(r)\, u_k^{(0)}(r)\, u_k(r). \tag{A5}$$

Expression (A5) can be integrated to obtain the Wronskian identity,

$$W\left(u_k^{(0)}, u_k\right)\Big|_{r_c}^{R} = 2\mu \int_{r_c}^{R} dr\, V^{(1)}(r)\, u_k^{(0)}(r)\, u_k(r), \tag{A6}$$

where $W(f, g) = f(r)g'(r) - f'(r)g(r)$ is the Wronskian, and r_c and R are, respectively, the ultraviolet and infrared cutoffs. The infrared cutoff R can be eliminated by taking into account the long-distance behavior of the $u_k^{(0)}$ and u_k reduced wave functions, which is given by

$$u_k^{(0)}(r) \overset{r\to\infty}{\longrightarrow} \frac{1}{\mathcal{A}^{(0)}(k)} \frac{\sin(k\,r + \delta^{(0)})}{\sin\delta^{(0)}}, \tag{A7}$$

$$u_k(r) \overset{r\to\infty}{\longrightarrow} \frac{1}{\mathcal{A}(k)} \frac{\sin(k\,r + \delta)}{\sin\delta}, \tag{A8}$$

where $\mathcal{A}^{(0)}(k)$ and $\mathcal{A}(k)$ are normalization factors that ensure an energy-independent normalization of the reduced wave functions at the cutoff radius. With the previous wave functions the Wronskian can be evaluated at $R \to \infty$, resulting in

$$W\left(u_k^{(0)}, u_k^{(0+1)}\right)\Big|_R = -\frac{k}{\mathcal{A}^{(0)}\,\mathcal{A}} \frac{\sin(\delta - \delta^{(0)})}{\sin\delta\,\sin\delta^{(0)}}. \tag{A9}$$

Therefore, we arrive at the expression

$$\frac{k}{\mathcal{A}^{(0)}\mathcal{A}}\frac{\sin(\delta-\delta^{(0)})}{\sin\delta\sin\delta^{(0)}}+f(r_c) = -2\mu\int_{r_c}^{\infty}dr\,V^{(1)}(r)u_k^{(0)}(r)u_k(r), \quad \text{(A10)}$$

where $f(r_c)$ is just the Wronskian evaluated at $r=r_c$, that is, $f(r_c)=W(u_k^{(0)},u_k)|_{r_c}$, which does not depend on the momentum k as a consequence of the energy-independent normalization at $r=r_c$. The perturbative expansion of the previous formula can be obtained from the corresponding one of its components,

$$\delta(k)=\delta^{(0)}(k)+\delta^{(1)}(k)+\mathcal{O}((V^{(1)})^2), \quad \text{(A11)}$$
$$u_k(r)=u_k^{(0)}(r)+u_k^{(1)}(r)+\mathcal{O}((V^{(1)})^2), \quad \text{(A12)}$$
$$\mathcal{A}(k)=\mathcal{A}^{(0)}(k)+\mathcal{A}^{(1)}(k)+\mathcal{O}((V^{(1)})^2), \quad \text{(A13)}$$
$$f(r_c)=f^{(1)}(r_c)+\mathcal{O}((V^{(1)})^2), \quad \text{(A14)}$$

yielding the DWBA formula for the phase shift,

$$\frac{k}{\mathcal{A}^{(0)^2}}\frac{\delta^{(1)}(k;r_c)}{\sin\delta^{(0)^2}}+f^{(1)}(r_c)=-2\mu\int_{r_c}^{\infty}dr\,V^{(1)}(r)\,u_k^{(0)^2}(r), \quad \text{(A15)}$$

where the Wronskian term $f^{(1)}$ can be safely ignored in renormalized computations, as it vanishes once the first subtraction is done.

APPENDIX B: DWBA FOR THE TRIPLET CHANNEL

In this appendix we present the perturbative distorted-wave formulas for the phase shifts in the $^3S_1$-$^3D_1$ triplet channel. For that, we express the phase shifts as the expansion

$$\delta_\alpha(k;r_c)=\delta_\alpha^{(0)}+\delta_\alpha^{(2)}+\delta_\alpha^{(3)}+\mathcal{O}(Q^4), \quad \text{(B1)}$$
$$\delta_\beta(k;r_c)=\delta_\beta^{(0)}+\delta_\beta^{(2)}+\delta_\beta^{(3)}+\mathcal{O}(Q^4), \quad \text{(B2)}$$
$$\epsilon(k;r_c)=\epsilon^{(0)}+\epsilon^{(2)}+\epsilon^{(3)}+\mathcal{O}(Q^4), \quad \text{(B3)}$$

where we have chosen the eigen parametrization of the phase shifts [66], because in this parametrization the DWBA formulas take their simplest form. The expansion of the nuclear bar phase shifts [67] can be obtained from the previous one by re-expanding the relationships

$$\bar\delta_1+\bar\delta_2=\delta_\alpha+\delta_\beta, \quad \text{(B4)}$$
$$\sin(\bar\delta_1-\bar\delta_2)=\frac{\tan 2\bar\epsilon}{\tan 2\epsilon}, \quad \text{(B5)}$$
$$\sin(\delta_\alpha-\delta_\beta)=\frac{\sin 2\bar\epsilon}{\sin 2\epsilon}, \quad \text{(B6)}$$

according to the counting. The LO phase shifts, $\delta_\alpha^{(0)}$, $\delta_\beta^{(0)}$, and $\epsilon^{(0)}$, are obtained by solving nonperturbatively the OPE potential with one counterterm, which is used for fixing the triplet scattering length to $a_t=5.419$ fm. The exact procedure is explained in Ref. [61]. The expressions for the perturbative corrections to the LO phase shifts are the following:

$$\frac{\delta_\alpha^{(\nu)}(k;r_c)}{\sin^2\delta_\alpha^{(0)}}=-\frac{2\mu}{k}\,\mathcal{A}_\alpha^{(0)^2}(k)\,I_{\alpha\alpha}^{(\nu)}(k;r_c), \quad \text{(B7)}$$
$$\frac{\delta_\beta^{(\nu)}(k;r_c)}{\sin^2\delta_\beta^{(0)}}=-\frac{2\mu}{k^5}\,\mathcal{A}_\beta^{(0)^2}(k)\,I_{\beta\beta}^{(\nu)}(k;r_c), \quad \text{(B8)}$$
$$\epsilon^{(\nu)}(k;r_c)=-\frac{2\mu}{k^3}\,\frac{\mathcal{A}_\beta^{(0)}(k)\,\mathcal{A}_\alpha^{(0)}(k)}{\cot\delta_\beta^{(0)}-\cot\delta_\alpha^{(0)}}\,I_{\beta\alpha}^{(\nu)}(k;r_c), \quad \text{(B9)}$$

where the perturbative integrals $I_{\alpha\alpha}$, $I_{\beta\alpha}$, and $I_{\beta\beta}$ are defined as

$$I_{\rho\sigma}^{(\nu)}(k;r_c)=\int_{r_c}^{\infty}dr\Big[V_{\rm ss}^{(\nu)}(r)\,u_{k,\rho}^{(0)}(r)\,u_{k,\sigma}^{(0)}(r) + V_{\rm sd}^{(\nu)}(r)\left(u_{k,\rho}^{(0)}(r)\,w_{k,\sigma}^{(0)}(r)+\,w_{k,\rho}^{(0)}(r)\,u_{k,\sigma}^{(0)}(r)\right) + V_{\rm dd}^{(\nu)}(r)\,w_{k,\rho}^{(0)}(r)\,w_{k,\sigma}^{(0)}(r)\Big], \quad \text{(B10)}$$

with $\rho,\sigma=\alpha,\beta$. As in the singlet case, μ represents the reduced mass of the system, $u_{k,\alpha(\beta)}^{(0)}$ and $w_{k,\alpha(\beta)}^{(0)}$ are the LO s- and d-wave reduced wave functions for the $\alpha(\beta)$ scattering states in an energy-independent normalization at the origin/cutoff radius, and $\mathcal{A}_\alpha^{(0)}$ and $\mathcal{A}_\beta^{(0)}$ are the normalization factors that ensure the previous condition. The asymptotic normalization of the α and β scattering states is taken to be

$$\mathcal{A}_\alpha^{(0)}\,u_{k,\alpha}^{(0)}(r)\to\cos\epsilon^{(0)}\left(\cot\delta_\alpha^{(0)}\,\hat j_0(kr)-\hat y_0(kr)\right),$$
$$\mathcal{A}_\alpha^{(0)}\,w_{k,\alpha}^{(0)}(r)\to\sin\epsilon^{(0)}\left(\cot\delta_\alpha^{(0)}\,\hat j_2(kr)-\hat y_2(kr)\right), \quad \text{(B11)}$$
$$k^2\,\mathcal{A}_\beta^{(0)}\,u_{k,\beta}^{(0)}(r)\to-\sin\epsilon^{(0)}\left(\cot\delta_\beta^{(0)}\,\hat j_0(kr)-\hat y_0(kr)\right),$$
$$k^2\,\mathcal{A}_\beta^{(0)}\,w_{k,\beta}^{(0)}(r)\to\cos\epsilon^{(0)}\left(\cot\delta_\beta^{(0)}\,\hat j_2(kr)-\hat y_2(kr)\right), \quad \text{(B12)}$$

with $\hat j_l(x)=x\,j_l(x)$, $\hat y_l(x)=x\,y_l(x)$, where $j_l(x)$ and $y_l(x)$ are the regular and irregular spherical Bessel functions. Owing to the energy-independent normalization of the wave functions at the cutoff radius, they can be expanded at short distances as [61]

$$u_{k,\alpha(\beta)}^{(0)}(r)=\sum_{n=0}^{\infty}u_{2n,\alpha(\beta)}^{(0)}(r)k^{2n}, \quad \text{(B13)}$$

where the behavior is given by

$$u_{2n,\alpha(\beta)}^{(0)}(r)\sim r^{3/4+5n/2}\,f\left(\sqrt{\frac{a}{r}}\right), \quad \text{(B14)}$$

with $f(x)$ some combination of $\sin x$, $\cos x$, and $e^{-\sqrt{2}x}$. The length scale a is related to the strength of the tensor force. In principle, the general solution of the Schrödinger equation for the tensor OPE potential also admits an $e^{+\sqrt{2}x}$ component that would destroy the renormalizability of the theory, as it generates divergences that cannot be absorbed by any finite number of counterterms. The previous component does not appear, however, if the LO wave functions have been properly renormalized. Therefore, what is essential is the power law behavior of the wave functions, which dictates the divergence

structure of the perturbative integrals $I^{(\nu)}_{\rho\sigma}$:

$$I^{(\nu)}_{\rho\sigma}(k;r_c) = I^{(\nu)}_{0,\rho\sigma}(r_c) + k^2\, I^{(\nu)}_{2,\rho\sigma}(r_c) + I^{(\nu)}_{R,\rho\sigma}(k;r_c), \quad \text{(B15)}$$

with $I^{(\nu)}_{0,\rho\sigma}$ and $I^{(\nu)}_{2,\rho\sigma}$ the divergent pieces of the integral and $I^{(\nu)}_{R,\rho\sigma}$ the regular piece. We can regularize the integral $I^{(\nu)}_{\rho\sigma}$ by including two free parameters,

$$\hat{I}^{(\nu)}_{\rho\sigma}(k;r_c) = \lambda^{(\nu)}_{0,\rho\sigma} + \lambda^{(\nu)}_{2,\rho\sigma}k^2 + I^{(\nu)}_{\rho\sigma}(k;r_c), \quad \text{(B16)}$$

which are to be fitted to the scattering data of the corresponding phase. The previous procedure yields a total of six counterterms for regularizing the NLO and N^{2}LO phase shifts. As in the singlet case, the finite piece of one of these parameters ($\lambda^{(\nu)}_{0,\alpha\alpha}$) is redundant, as it only affects the triplet scattering length a_t, which was already fixed at LO. In other words, six data are enough to determine the NLO/N^{2}LO phase shifts in the triplet.

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